

# Direct Measurement of the Elastohydrodynamic Lift Force at the Nanoscale – Supplementary Information –

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## SOFT LUBRICATION MODEL

We consider a sphere of radius  $R$ , immersed in a Newtonian fluid of viscosity  $\eta$  and density  $\rho$  (see Fig. 1 in the article). The sphere is placed at a distance  $d$  from a thick incompressible elastic substrate of shear modulus  $G$ . The substrate is translated horizontally at an oscillatory velocity  $\vec{V}(t) = A\omega \sin(\omega t)\vec{e}_x$ , where  $t$  is time,  $\vec{e}_x$  is the unit vector along the  $x$ -axis, and  $A$  and  $\omega$  are respectively the amplitude and angular frequency imposed by the piezo stage. We focus on the regime where the gap distance  $d$  is much smaller than the sphere radius  $R$ , such that the lubrication scale separation can be applied. Besides, the Reynolds number  $\text{Re} = \rho A\omega\sqrt{Rd}/\eta$  is much smaller than unity in all experiments, such that we can neglect inertial terms. We also neglect the time dependence of the gap distance since the vertical velocity of the piezo is much smaller than the horizontal one. In addition, the cantilever's stiffness is large enough, so that the cantilever's deflection amplitude is negligible with respect to the gap distance in the large-gap-distance limit. The resulting horizontal fluid velocity field  $\vec{v}(\vec{r}, z, t)$  is a linear combination of Poiseuille and Couette flows:

$$\vec{v}(\vec{r}, z, t) = \frac{1}{2\eta}\vec{\nabla}p\left[z^2 - z(h_0 + \delta) + h_0\delta\right] + \frac{h_0 - z}{h_0 - \delta}\vec{V}, \quad (\text{S1})$$

where  $z$  is the vertical coordinate,  $\vec{r} = (x, y)$  the horizontal ones,  $h_0(\vec{r}) \simeq d + \frac{r^2}{2R}$  is the gap profile in the absence of any substrate's deformation and invoking the parabolic approximation for the sphere in the contact zone, and  $\delta(\vec{r}, t)$  is the substrate's deformation. The excess pressure field with respect to the ambient is denoted  $p(\vec{r}, t)$ . A polar coordinate system  $\vec{r} = (r, \theta)$  is used, with  $\theta$  being the angle with respect to the  $x$ -axis. Expressing volume conservation, one can show that the fluid thickness,  $h = h_0 - \delta$ , follows the lubrication Reynolds equation:

$$\partial_t h(\vec{r}, t) = \vec{\nabla} \cdot \left[ \frac{h^3(\vec{r}, t)}{12\eta}\vec{\nabla}p(\vec{r}, t) - \frac{h(\vec{r}, t)}{2}\vec{V}(t) \right]. \quad (\text{S2})$$

We assume that the elastic response is linear, and that the incompressible substrate is semi-infinite as the contact length  $\ell = \sqrt{2Rd}$  is much smaller than the substrate's thickness. Thus, the deformation reads:

$$\delta(\vec{r}, t) = -\frac{1}{4\pi G} \int d^2\vec{r}' \frac{p(\vec{r}', t)}{|\vec{r} - \vec{r}'|}. \quad (\text{S3})$$

In the following, we non-dimensionalize all the variables through the lubrication scaling:

$$h(\vec{r}, t) = d\hat{h}(\hat{\vec{r}}, \hat{t}), \quad \vec{r} = \ell\hat{\vec{r}}, \quad \delta(\vec{r}, t) = d\hat{\delta}(\hat{\vec{r}}, \hat{t}), \quad (\text{S4})$$

$$p(\vec{r}, t) = \frac{\eta A\omega\ell}{d^2}\hat{p}(\hat{\vec{r}}, \hat{t}), \quad t = \frac{\ell}{A\omega}\hat{t}, \quad \vec{v}(\vec{r}, t) = A\omega\hat{v}(\hat{\vec{r}}, \hat{t}), \quad \vec{V}(t) = A\omega\hat{V}(\hat{t})\vec{e}_x, \quad (\text{S5})$$

with  $\hat{V}(\hat{t}) = \sin(\ell\hat{t}/A)$ . For clarity, we further drop the hat symbols ( $\cdot$ ). The governing equations are then:

$$12\partial_t h(\vec{r}, t) = \vec{\nabla} \cdot \left[ h^3(\vec{r}, t)\vec{\nabla}p(\vec{r}, t) - 6h(\vec{r}, t)\vec{V}(t) \right], \quad (\text{S6})$$

$$h(\vec{r}, t) = 1 + r^2 - \delta(\vec{r}, t), \quad (\text{S7})$$

and:

$$\delta(\vec{r}, t) = -\frac{\xi}{2\pi} \int_{\mathbb{R}^2} d^2\vec{r}' \frac{p(\vec{r}', t)}{|\vec{r} - \vec{r}'|}, \quad (\text{S8})$$

with  $\xi = \eta A \omega R / (G d^2) = \sqrt{2} \kappa$ , where  $\kappa$  is the dimensionless compliance introduced in the article, that corresponds to the typical substrate's deformation over the gap distance in the limit of small  $\kappa$ .

### PERTURBATION THEORY

Assuming that the deformation of the substrate is small with respect to the typical gap distance, *i.e.*  $\xi \ll 1$ , we perform a perturbative expansion in fluid thickness profile and pressure, as:

$$h(\vec{r}, t) = h_0(\vec{r}) + \xi h_1(\vec{r}, t) + \mathcal{O}(\xi^2), \quad (\text{S9})$$

$$p(\vec{r}, t) = p_0(\vec{r}, t) + \xi p_1(\vec{r}, t) + \mathcal{O}(\xi^2), \quad (\text{S10})$$

where the subscript 0 corresponds to the case of a rigid wall, for which  $h_0(\vec{r}) = 1 + r^2$  in particular. Equation (S6) gives at zeroth order:

$$0 = \vec{\nabla} \cdot \left( h_0^3 \vec{\nabla} p_0 - 6h_0 \vec{V} \right), \quad (\text{S11})$$

and thus, in polar coordinates:

$$r^2 \partial_r^2 p_0 + \left( r + \frac{6r^3}{1+r^2} \right) \partial_r p_0 + \partial_\theta^2 p_0 = \frac{12r^3 V \cos \theta}{(1+r^2)^3}. \quad (\text{S12})$$

We solve this equation using an angular mode decomposition:

$$p_0(r, \theta, t) = P_0^{(1)}(r, t) \cos \theta, \quad (\text{S13})$$

where the amplitude  $P_0^{(1)}$  satisfies the ordinary differential equation:

$$r^2 \partial_r^2 P_0^{(1)} + \left( r + \frac{6r^3}{1+r^2} \right) \partial_r P_0^{(1)} - P_0^{(1)} = \frac{12r^3 V}{(1+r^2)^3}. \quad (\text{S14})$$

Imposing the natural boundary conditions,  $p_0(r \rightarrow \infty, \theta, t) = 0$  and  $p_0(r = 0, \theta, t) < \infty$ , the zeroth-order solution reads:

$$p_0(r, \theta, t) = -\frac{6rV(t) \cos \theta}{5(1+r^2)^2}. \quad (\text{S15})$$

The first-order correction  $h_1$  in fluid thickness profile can then be computed by inserting  $p_0$  into the first-order version of Eq. (S8), as:

$$h_1(\vec{r}, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2\vec{r}' \frac{p_0(\vec{r}', t)}{|\vec{r} - \vec{r}'|}. \quad (\text{S16})$$

We solve this convolution product by introducing the spatial Fourier transform  $\tilde{h}_1(\vec{k}, t) = \int_{\mathbb{R}^2} d^2\vec{r} h_1(\vec{r}, t) e^{-i\vec{k}\vec{r}}$ . In Fourier space, Eq. (S16) becomes:

$$\tilde{h}_1(\vec{k}, t) = \frac{\tilde{p}_0(\vec{k}, t)}{k}. \quad (\text{S17})$$

One can then calculate and invert this Fourier transform, and get:

$$h_1(r, \theta, t) = -\frac{3V(t)}{5r} \left[ \mathcal{K}(-r^2) - \frac{\mathcal{E}(-r^2)}{1+r^2} \right] \cos \theta, \quad (\text{S18})$$

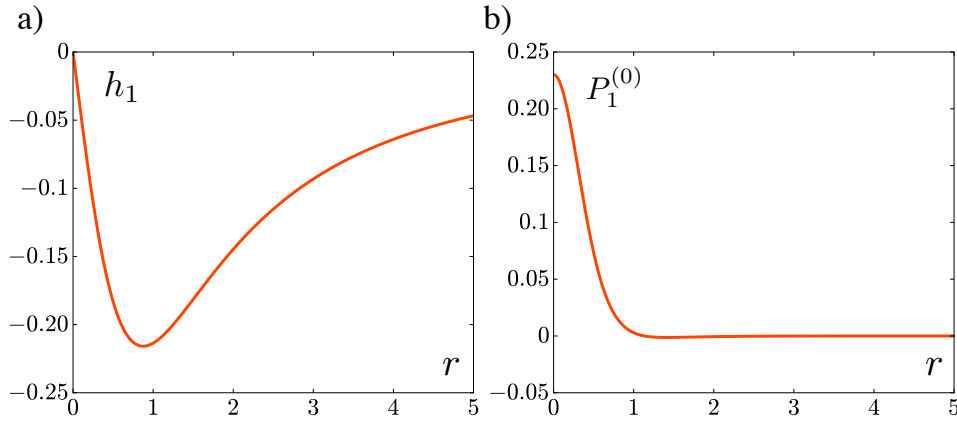


FIG. S1: a) First-order correction  $h_1$  in fluid thickness as a function of radial coordinate  $r$ , for the particular case where  $V(t) = 1$  and  $\theta = 0$ , as obtained from Eq. (S18). b) Isotropic part  $P_1^{(0)}$  of the first-order correction  $p_1$  in pressure, for the particular case where  $V(t) = 1$ , as obtained from a numerical integration of Eq. (S21).

where  $\mathcal{K}$  and  $\mathcal{E}$  are the complete elliptic integrals of the first and second kinds. That solution is plotted in Fig. S1a) for the particular case where  $V(t) = 1$  and  $\theta = 0$ . Besides, the first-order correction  $p_1$  satisfies the first-order version of Eq. (S6):

$$12\partial_t h_1 = \vec{\nabla} \cdot \left( h_0^3 \vec{\nabla} p_1 + 3h_0^2 h_1 \vec{\nabla} p_0 - 6h_1 \vec{V} \right), \quad (\text{S19})$$

where everything is now known apart from  $p_1$ . After some algebra, the latter equation can be expressed in polar coordinates:

$$\begin{aligned} r^2 \partial_r^2 p_1 + \left( r + \frac{6r^3}{1+r^2} \right) \partial_r p_1 + \partial_\theta^2 p_1 = & \frac{36 r^2 V^2(t)}{25(1+r^2)^6} \left[ (-10 + 2r^2) \mathcal{E}(-r^2) + (8 + 7r^2 - r^4) \mathcal{K}(-r^2) \right] \\ & - \frac{36 r \dot{V}(t)}{5(1+r^2)^4} \left[ -\mathcal{E}(-r^2) + (1+r^2) \mathcal{K}(-r^2) \right] \cos \theta \\ & + \frac{36 V^2(t)}{25(1+r^2)^6} \left[ -2(2 + 9r^2 + r^4) \mathcal{E}(-r^2) + (1+r^2)(4 + 16r^2 + 3r^4) \mathcal{K}(-r^2) \right] \cos(2\theta). \end{aligned} \quad (\text{S20})$$

We invoke an angular mode decomposition, through  $p_1(r, \theta, t) = P_1^{(0)}(r, t) + P_1^{(1)}(r, t) \cos \theta + P_1^{(2)}(r, t) \cos(2\theta)$ , and get in particular the ordinary differential equation for the isotropic term:

$$r^2 \partial_r^2 P_1^{(0)} + \left( r + \frac{6r^3}{1+r^2} \right) \partial_r P_1^{(0)} = \frac{36 r^2 V^2(t)}{25(1+r^2)^6} \left[ (-10 + 2r^2) \mathcal{E}(-r^2) + (8 + 7r^2 - r^4) \mathcal{K}(-r^2) \right]. \quad (\text{S21})$$

This equation can then be solved numerically, using *e.g.* an order-4 Runge-Kutta algorithm, together with a shooting parameter that converges when  $P_1^{(0)} \rightarrow 0$  at large  $r$ . The solution is shown in Fig. S1(b) for the particular case where  $V(t) = 1$ . Note that, by linearity of the ordinary differential equation on  $P_1^{(0)}$ , the solution for an arbitrary  $V(t)$  is immediately obtained from this particular solution through a simple multiplication by  $V^2(t)$ . Finally, the normal (“lift”) force is defined as  $\int_{\mathbb{R}^2} d^2 \vec{r} p(\vec{r}, t)$ , and can thus be computed at first order in  $\xi$  from the results above. We stress that we focused only on the isotropic angular mode of  $p_1$ , as the other ones do not contribute to the force, due to the angular integration. Putting back dimensions, this gives the first-order expression of the lift force  $F_{\text{lift}}(t)$ :

$$F_{\text{lift}}(t) \simeq 8\pi F^* \xi \int_{\mathbb{R}_+} \hat{r} d\hat{r} \hat{P}_1^{(0)}(\hat{r}, \hat{t}) \approx 0.416 \frac{\eta^2 V^2(t)}{G} \left( \frac{R}{d} \right)^{5/2}, \quad (\text{S22})$$

where  $F^* = \eta A \omega R^{3/2} / (2d)^{1/2}$  is the typical lubrication force scale defined in the article. Therefore, the temporal average  $F$  of  $F_{\text{lift}}$  over a period  $2\pi/\omega$  of oscillation is given by:

$$F \approx 0.416 \kappa F^*. \quad (\text{S23})$$

Interestingly, Eq. (S22) is not restricted to constant  $V$ , provided no new physics (*e.g.* viscoelasticity, poroelasticity, acoustic waves, etc.) is added due to the temporal variation of  $V(t)$ . However, the quasi-steady form of Eq. (S22) – where the time dependence is solely implicit through  $V(t)$  – imply that transient effects do not contribute to the average lift force  $F$ , at first order in  $\kappa$  (or  $\xi$ ). For this reason, we simplified the discussion in the article by considering only an analogous steady problem, with the root-mean-squared value  $A\omega/\sqrt{2}$  replacing the steady velocity all along the article. Nevertheless, it is generally expected that transient terms will become relevant at higher orders in  $\kappa$  (or  $\xi$ ).

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